ON CERTAIN CASES OF INTEGRABILITY OF THE EQUATIONS OF MOTION OF A PARTICLE UNDER THE ACTION OF A NEWTONIAN FORCE AND ADDITIONAL PERTURBATION FORCES

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PMM Vol.27, No.2, 1963, pp.211-217

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(Received November 20, 1962)

The derivation of the equations of motion. Consider the equation of motion of a particle M with respect to a fixed origin O

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{k\mathbf{r}}{r^3} + \mathbf{F}, \qquad k = \mathrm{const}, \qquad r = |\mathbf{r}| \tag{1.1}$$

with the initial conditions

$$\mathbf{r} = \mathbf{r}_0, \qquad \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}_0, \qquad t = t_0 \tag{1.2}$$

Here, $\mathbf{r}$ is the position vector of the particle M, $\mathbf{F}$ is a force acting on a unit mass in addition to a Newtonian force $-kr^{-3}\mathbf{r}$. Let us introduce the designations

$$\mathbf{G} = \mathbf{r} \times \frac{d\mathbf{r}}{dt}, \qquad |\mathbf{G}| = G, \qquad \boldsymbol{\gamma} = \frac{\Lambda}{r}\mathbf{r}, \qquad |\boldsymbol{\gamma}| = 1 \tag{1.3}$$

Here, $\mathbf{G}$ is the moment of momentum of the particle M whose mass is equal to unity; γ is a unit vector directed from the origin O towards the particle M.

Taking the derivative of $\mathbf{G}$ (1.3) with respect to t, we obtain from (1.1)

$$\frac{d\mathbf{G}}{dt} = \mathbf{r} \times \mathbf{F} \tag{1.4}$$

Substituting $\mathbf{r} = \gamma r$ into (1.1) and taking a scalar product of the equation (1.1) and γ we obtain the equations for r

$$\frac{d^2r}{dt^2} - \frac{1}{r^3}G^2 = -\frac{k}{r^3} + \mathbf{F}\boldsymbol{\gamma} \tag{1.5}$$

Transforming the vector product of the vectors $\mathbf{G}$, γ, we obtain the differential equation for γ

$$\frac{d\boldsymbol{\gamma}}{dt} = r^{-2}\mathbf{G}\times\boldsymbol{\gamma} \tag{1.6}$$

The system of equations (1.4), (1.5), (1.6) with two known integrals

$$\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1, \qquad \boldsymbol{\gamma}\cdot\mathbf{G} = 0 \tag{1.7}$$

is equivalent to the equation (1.1). Let us make a substitution of the dependent variable r and the independent variable t

$$u = r^{-1}, \qquad d\tau = r^{-2}\,dt = u^2\,dt \tag{1.8}$$

The equations (1.4) to (1.6) take the form

$$\frac{d\mathbf{G}}{d\tau} = u^{-3}\boldsymbol{\gamma}\times\mathbf{F}, \qquad \frac{d\boldsymbol{\gamma}}{d\tau} = \mathbf{G}\times\boldsymbol{\gamma}, \qquad \frac{d^2u}{d\tau^2} + G^2u = k - u^{-2}\mathbf{F}\cdot\boldsymbol{\gamma} \tag{1.9}$$

Let us form at point O an orthogonal vectorial trihedron consisting of the unit vectors γ, $\mathbf{g}$ and $\mathbf{s}$, where

$$\mathbf{g} = G^{-1}\mathbf{G}, \qquad \mathbf{s} = \mathbf{g}\times\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma} = r^{-1}\mathbf{r} \tag{1.10}$$

We will call the plane passing through the origin O and the moving particle M, and containing the velocity vector $d\mathbf{r}/dt$, the plane of motion.

The vector $\mathbf{g}$ is perpendicular to the plane of motion, vector $\mathbf{s}$ lies in the plane of motion.

We designate the projections of the perturbation force $\mathbf{F}$ in the directions of the vectors γ, $\mathbf{g}$ and $\mathbf{s}$, by F_γ, F_g and F_s

$$\mathbf{F} = F_\gamma\boldsymbol{\gamma} + F_g\mathbf{g} + F_s\mathbf{s} \tag{1.11}$$

Let us simplify the first of equations (1.9) by forming a separate equation for G and $\mathbf{g}$; substituting in (1.9) $\mathbf{G} = G\mathbf{g}$ and $\mathbf{F}$ according to (1.11) we obtain the equations

$$\frac{dG}{d\tau} = \frac{1}{u^3}F_s, \qquad \frac{d\mathbf{g}}{d\tau} = \frac{F_g}{u^3G}\boldsymbol{\gamma}\times\mathbf{g} \tag{1.12}$$

Now we introduce a new auxiliary vector $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = G\mathbf{g} + u^{-3}G^{-1}F_g\boldsymbol{\gamma} \tag{1.13}$$

From the equation (1.9) for γ and from (1.12) for $\mathbf{g}$ we can write

$$\frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\omega} \times \boldsymbol{\gamma}, \qquad \frac{d\mathbf{g}}{d\tau} = \boldsymbol{\omega} \times \mathbf{g} \tag{1.14}$$

that is vector $\boldsymbol{\omega}$ is the angular velocity vector (see, for instance [1, p.64]) of motion of the vectorial trihedron γ, $\mathbf{g}$ and $\mathbf{s}$. If $F_g \equiv 0$, then $\mathbf{g}$ = const and the motion takes place in a fixed plane.

The equations (1.14) and (1.9) for u and the equations (1.12) for G form a closed system. It is of the ninth order, but three integrals are known in advance

$$\boldsymbol{\gamma\gamma} = 1, \qquad \mathbf{gg} = 1, \qquad \boldsymbol{\gamma}\mathbf{g} = 0 \tag{1.15}$$

The obtained system of equations is equivalent to the equation (1.1). Its deficiency is that it is of a higher order than (1.1); its merit is that for $\mathbf{F} \equiv 0$ it becomes a system of differential equations with constant coefficients. Therefore, for sufficiently small values of the force $F = |\mathbf{F}|$ the method of a small parameter can be applied. Another property of the system (1.5), (1.14) and the first equation (1.12) is that if the forces F_γ and F_s, depend only on u, τ and G, the equations (1.5) and (1.12) can be integrated independently of equations (1.14). The integration of equations (1.14) is equivalent to the problem of finding the position of a body from the known vector of angular velocity $\boldsymbol{\omega}$ in the moving coordinate axes. This problem is considered in [1, pp.100-136] and can be reduced to the integration of a single nonlinear complex Darboux equation (see, for instance, [1, p.130]).

If the plane of motion is invariable, i.e. $\mathbf{g}$ = const, then from (1.12) for $\mathbf{g}$ we obtain $F_g = 0$. In this case the difficulty of solving the problem lies entirely in the integration of the equations (1.9) for u and the equation (1.12) for G.

The plane of motion moves in a regular precession in the case when

$$F_s \equiv 0, \quad F_g = au^3 = ar^{-3}, \quad a = \text{const} \tag{1.16}$$

From the equation (1.12) we have G = const, $\boldsymbol{\omega}$ = const in the moving coordinate axes and consequently, in the fixed coordinate axes as well.

A.I. Lur'e has indicated to the author that the results of [2] related to the motion of a satellite in a circular orbit (u = const) with the forces F_g = const, $F_s = F_\gamma = 0$, can be obtained in a considerably simpler way from the equations, analogous to (1.14).

Finally, let us point out that the obtained equations are particularly convenient for the study of nearly circular orbits, whereas the classical equations (see, for instance, [3, p.350]) are not suitable for small values of the eccentricity e of the osculating ellipse.

2. The case of plane motion. If $F_g \equiv 0$ the plane of motion is fixed. Let us make a substitution in (1.5) and (1.12) for G, changing the independent variable τ to φ and the dependent variable G to p by the formulas

$$d\varphi = G\,d\tau = Gu^2\,dt, \qquad p = G^2 \tag{2.1}$$

As a result we obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{k}{p} - \frac{1}{u^2p}F_\gamma - \frac{1}{u^3p}F_s\frac{du}{d\varphi}, \qquad \frac{dp}{d\varphi} = \frac{2F_s}{u^3} \tag{2.2}$$

For $F_g \equiv 0$ the variable φ has a definite geometrical meaning; it is an angle with the vertex at the origin O, which is described by the position vector $\mathbf{r}$ on the fixed plane of motion.

Let us assume that in (2.2) the forces F_γ, F_s do not depend on φ, in which case the order of the system can be reduced. We introduce u as the independent variable and $q = du/d\varphi$ as the dependent variable. Then we have

$$q\frac{dq}{du} = -u + \frac{1}{p}\left(k - \frac{1}{u^2}F_\gamma - \frac{1}{u^3}F_sq\right), \qquad q\frac{dp}{du} = \frac{2}{u^3}F_s \qquad \left(q = \frac{du}{d\varphi}\right) \tag{2.3}$$

The system of equations (2.3) can be integrated if the external forces have the form

$$F_\gamma = ku^2 + au^3, \qquad F_s = bu^4, \qquad F_g = 0 \quad (a, b = \text{const}) \tag{2.4}$$

since, dividing the first equation (2.3) by the second, we obtain an equation which does not contain u and is linear with respect to q.

3. The integration in a particular case of external forces. Let us consider a case of additional forces, more general than (2.4)

$$F_\gamma = \frac{k}{r^2} + \frac{a(G)}{r^3}, \qquad F_s = \frac{b(G)}{r^4}, \qquad F_g = 0 \tag{3.1}$$

The expression (3.1) for the force F indicates that, in fact, the

Newtonian force is excluded from consideration. The particle M is acted upon by a force inversely proportional to the third power of the distance r in the direction of the vector $\mathbf{r}$, and a force inversely proportional to the fourth power of the distance r in the direction of the vector $\mathbf{s}$, which lies in the plane of motion perpendicularly to r.

If a = const, b = const and, at the initial instant t_0, a is chosen so as to have $F_\gamma = 0$, i.e. $a = -kr_0$, then the maximal additional force F_γ in the direction of r must be attained for

$$r = 1.5r_0, \qquad F_\gamma(1.5r_0) = \frac{4}{27}\frac{k}{r_0^2} \tag{3.2}$$

Here kr_0^{-2} is a Newtonian force of attraction for $r = r_0$.

The equations (1.5) and (1.19) with (3.1) take the form

$$\frac{d^2u}{d\tau^2} + G^2u = -a(G)\,u, \qquad \frac{dG}{d\tau} = b(G)\,u \tag{3.3}$$

Eliminating u from the first equation by means of the second, we arrive at an integrable equation of the third order

$$\frac{d}{d\tau}\left[\frac{d}{d\tau}\left(\frac{1}{b(G)}\frac{dG}{d\tau}\right)\right] + \frac{G^2}{b(G)}\frac{dG}{d\tau} + \frac{a(G)}{b(G)}\frac{dG}{d\tau} = 0 \tag{3.4}$$

We integrate the equation (3.4) once and multiply by $2b^{-1}(G)\,dG/d\tau$. Thus we arrive at a new equation of the second order

$$\frac{2}{b(G)}\frac{dG}{d\tau}\frac{d}{d\tau}\left[\frac{1}{b(G)}\frac{dG}{d\tau}\right] + \frac{2\psi_1(G)}{b(G)}\frac{dG}{d\tau} = 0 \tag{3.5}$$

$$\psi_1(G) = \int (G^2 + a(G))\frac{dG}{b(G)} \tag{3.6}$$

Integrating the equation (3.5) with respect to τ, we once again arrive at an equation of the first order

$$\left[\frac{1}{b(G)}\frac{dG}{d\tau}\right]^2 = -\int \frac{2\psi_1(G)}{b(G)}\,dG \equiv \psi_2(G) \tag{3.7}$$

From the equation (3.3) we find

$$u = \sqrt{\psi_2(G)}, \qquad r = [\psi_2(G)]^{-1/2} \tag{3.8}$$

From (3.7) and (1.8) we find the expression for t

$$t = \int \frac{d\tau}{u^2} = \int \frac{dG}{b(G)\sqrt{\psi_2^3(G)}} \tag{3.9}$$

From (3.7), (2.1) we can find the expression for the angle φ

$$\varphi = \int \frac{G\,dG}{b\,(G)\,\sqrt{\psi_2\,(G)}} \tag{3.10}$$

We obtain the solution of the problem, where r, t and φ are given in terms of functions of the parameter G. The arbitrary constants can be found from the conditions (1.2). Let us state the final answer for the solution of the problem (1.1) with initial conditions (1.2) in the case of forces of the form (2.6).

The solution has been carried out by the method given in this article. The function $\psi_2(G)$ takes the form

$$\psi_2\,(G) = C_2 - \frac{1}{6b^2} G^4 - \frac{a}{b^2} G^2 + \frac{2}{b} C_1 G \tag{3.11}$$

$$C_1 = |\,\mathbf{r}_0\,|^{-1}\,(\mathbf{r}_0\dot{\mathbf{r}}_0) - \frac{1}{3b}\,|\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,|^3 - \frac{a}{b}\,|\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,| \tag{3.12}$$

$$C_2 = |\,\mathbf{r}_0\,|^{-2} + \frac{1}{6b^2}\,|\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,|^4 + \frac{a}{b^2}\,|\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,|^2 + \frac{2}{b} C_1\,|\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,|$$

For r, t and φ we find from (3.8) to (3.10)

$$t = t_0 + \int_{G_0}^{G} \frac{dG}{b\,(G)\,\sqrt{\psi_2^{\,3}\,(G)}}, \qquad r = \frac{1}{\sqrt{\psi_2\,(G)}} \tag{3.13}$$

$$\varphi = \varphi_0 + \int_{G_0}^{G} \frac{G\,dG}{b\,(G)\,\sqrt{\psi_2\,(G)}}, \qquad G_0 = |\,\mathbf{r} \times \mathbf{r}_0\,| \tag{3.14}$$

For r we obtain the expression

$$\mathbf{r} = r\,(\boldsymbol{\gamma}_0 \cos\varphi + \mathbf{s}_0 \sin\varphi) = r\left(\frac{\mathbf{r}_0}{|\,\mathbf{r}_0\,|} \cos\varphi + \frac{\dot{\mathbf{r}}_0\,(\mathbf{r}_0)^2 - \mathbf{r}_0\,(\mathbf{r}_0\dot{\mathbf{r}}_0)}{|\,\mathbf{r}_0\,| \cdot |\,\mathbf{r}_0 \times \dot{\mathbf{r}}_0\,|} \sin\varphi\right) \tag{3.15}$$

Since the expressions (3.1) for the forces F_γ, F_s contain arbitrary functions of G, the method of integration given in this article can be applied for an approximate numerical or graphical solution of the equations (1.10) and (1.12) when forces are given in some other manner. To this end we choose arbitrary $a(G)$ and $b(G)$ in (3.1). We find the expressions (3.8) to (3.10) for r, φ and t, and then we improve the functions $a(G)$ and $b(G)$. The proposed approach is more convenient than the direct integration of the equations (1.10) and (1.12), since it always requires the integration of known functions of G, rather than differential equations.

4. The case of integrability of the equations of motion when the additional force F is directed along the velocity dr/dt of the moving particle M. Let us designate by α the angle formed by the position vector $\mathbf{r}$ and the velocity vector $d\mathbf{r}/dt$

$$\cos\alpha = \mathbf{r}\frac{d\mathbf{r}}{dt}\left(|\mathbf{r}|\cdot\left|\frac{d\mathbf{r}}{dt}\right|\right)^{-1}, \qquad \tan\alpha = \frac{\mathbf{r}\,d\varphi}{dr} \tag{4.1}$$

If the force $\mathbf{F}$ is directed along the velocity, then

$$\mathbf{F}\parallel\frac{d\mathbf{r}}{dt}, \qquad F_{\gamma} = F\cos\alpha, \qquad F_{s} = F\sin\alpha, \qquad F = |\mathbf{F}| \tag{4.2}$$

From (4.1) and (4.2) we find

$$F_{\gamma} + \frac{1}{u}F_{s}\frac{du}{d\varphi} = F\cos\alpha - \frac{1}{r}\frac{dr}{d\varphi}F\sin\alpha \equiv 0 \tag{4.3}$$

For the force $\mathbf{F}$ directed along the velocity, the equations (2.2) and (2.3) take a relatively simple form

$$\frac{d^2u}{d\varphi^2} + u = \frac{k}{p}, \qquad \frac{dp}{d\varphi} = \frac{2F_s}{u^3} \tag{4.4}$$

It is interesting to note that the change of the radius r depends on the force $\mathbf{F}$, but only indirectly, by way of p.

a) Let the projection of the force F_s be given in the form

$$F_s = a(\varphi)\,b(p)\,u^3 \tag{4.5}$$

In this case the equations (4.4) can be integrated.

We will write down the solution for the particular case (4.5), when the force $\mathbf{F}$ is acting along the velocity and has the form

$$F = |\mathbf{F}| = \frac{1}{r^3}\frac{A(\varphi)}{\sin\alpha}, \qquad F_s = \frac{A(\varphi)}{r^3} = A(\varphi)\,u^3 \tag{4.6}$$

From (4.4), (1.2) we find the solution

$$p(\varphi) = |\mathbf{r}_0\times\dot{\mathbf{r}}_0|^2 + 2\int_{\varphi_0}^{\varphi} A(\varphi)\,d\varphi \tag{4.7}$$

$$r(\varphi) = \left(\frac{\cos\varphi}{|\mathbf{r}_0|} - \frac{\mathbf{r}_0\dot{\mathbf{r}}_0\sin\varphi}{|\mathbf{r}_0|\cdot|\mathbf{r}_0\times\dot{\mathbf{r}}_0|} + \int_{\varphi_0}^{\varphi}\frac{k\sin(\varphi-\tau)}{p(\tau)}\,d\tau\right)^{-1} \tag{4.8}$$

$$t = t_0 + \int_{\varphi_0}^{\varphi}\frac{r^2(\varphi)\,d\varphi}{\sqrt{p(\varphi)}} \tag{4.9}$$

The parameter is the angle φ which the position vector $\mathbf{r}$ describes in the plane of motion.

b) Let us change to a new independent variable u in (4.4) by setting

$$q = \frac{du}{d\varphi}, \qquad q = -\frac{1}{r^2}\frac{dr}{d\varphi} = -\frac{1}{r}\cot\alpha \tag{4.10}$$

The system of equations (2.4) and (2.5) takes the form

$$q\frac{dq}{du} = \frac{k}{p} - u, \qquad q\frac{dp}{du} = \frac{2F_s}{u^3} \tag{4.11}$$

The system of equations (4.11) will be integrable if in (4.11) we set

$$F_s = qa(u)\,b(p) = -ua(u)\,b(p)\cot\alpha \tag{4.12}$$

We will state the solution for the case when the force F has the form

$$F = A(r)\frac{\cos\alpha}{\sin^2\alpha} \tag{4.13}$$

From (4.11) we find the expressions for r, φ and t in terms of the parameter u

$$p(u) = |\mathbf{r}_0 \times \dot{\mathbf{r}}_0|^2 - 2\int_{u_0}^{u} \frac{1}{u} A\left(\frac{1}{u}\right) du, \qquad u_0 = |\mathbf{r}_0|^{-1} \tag{4.14}$$

$$q(u) = \left[\frac{(\mathbf{r}_0\dot{\mathbf{r}}_0)^2}{|\mathbf{r}_0|^2 \cdot |\mathbf{r}_0 \times \dot{\mathbf{r}}_0|^2} + 2\int_{u_0}^{u}\left(\frac{k}{p(u)} - u\right) du\right]^{1/2} \tag{4.15}$$

$$r = \frac{1}{u}, \qquad \varphi = \varphi_0 + \int_{u_0}^{u}\frac{du}{q(u)}, \qquad t = t_0 + \int_{u_0}^{u}\frac{du}{u^2 q(u)\sqrt{p(u)}} \tag{4.16}$$

c) From (4.1) we have

$$\cot\alpha = -rq, \qquad \sin\alpha = (1 + r^2q^2)^{-1/2} = u(u^2 + q^2)^{-1/2} \tag{4.17}$$

The equations (4.11) can be rewritten in the form

$$\frac{d(q^2+u^2)}{du} = \frac{2k}{p}, \qquad q\frac{dp}{du} = \frac{2F}{u^2\sqrt{u^2+q^2}} \tag{4.18}$$

For the independent variable we introduce the quantity μ

$$\mu = q^2 + u^2 \tag{4.19}$$

The quantity $(\mu)^{-1/2}$ has a definite geometrical meaning; it is the distance from the origin O to the straight line which passes through the particle M and is parallel to the velocity vector $d\mathbf{r}/dt$.

The equations (4.18) can be integrated if the expression for the force F has the form

$$F = |\mathbf{F}| = r^{-3} a(\mu) b(p) \cot \alpha \tag{4.20}$$

Let us state the final solution for the case of force $\mathbf{F}$ acting along the velocity vector $d\mathbf{r}/dt$ and having the form

$$F = a(\mu) \cot \alpha\, r^{-3} \tag{4.21}$$

From the equations (4.18) we find the expressions for r, φ and t in terms of the parameter μ

$$p(\mu) = |\mathbf{r}_0 \times \dot{\mathbf{r}}_0|^2 \exp\left\{-\int_{\mu_0}^{\mu} \frac{a(\mu)}{k\sqrt{\mu}} d\mu\right\}, \qquad \mu_0 = |\dot{\mathbf{r}}_0|^2 |\mathbf{r}_0 \times \dot{\mathbf{r}}_0|^{-2} \tag{4.22}$$

$$u(\mu) = \frac{1}{|\mathbf{r}_0|} + 0.5k^{-1} \int_{\mu_0}^{\mu} p(\mu)\, d\mu, \qquad r(\mu) = \frac{1}{u(\mu)} \tag{4.23}$$

$$\varphi = \varphi_0 + \frac{1}{2k} \int_{\mu_0}^{\mu} \frac{p(\mu)\, d\mu}{\sqrt{\mu - u^2(\mu)}}, \qquad t = t_0 + \frac{1}{2k} \int_{\mu_0}^{\mu} \frac{\sqrt{p(\mu)}\, d\mu}{u^2(\mu)\sqrt{\mu - u^2(\mu)}} \tag{4.24}$$

The equations (4.4) and (4.11) have a simple form and it appears that other ways of prescribing the force $\mathbf{F}$ can be found, for which these equations are integrable. In all the cases (a), (b) and (c) the expression for the force contains an arbitrary function of the parameter. All these cases can be utilized for approximate solutions when the force $\mathbf{F}$, directed along the velocity, is specified in some other manner. For instance, the cases (a) and (b) can be used in the study of the effect of friction, caused by the atmosphere, on satellites. Let us note that

$$\left|\frac{d\mathbf{r}}{dt}\right| = G\sqrt{u^2 + q^2} = \frac{G}{r \sin \alpha} \tag{4.25}$$

In conclusion the author wishes to thank A.I. Lur'e for discussion of this paper.

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Translated by O.S.